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Construction of Nonlinear Normal Modes by Shaw-Pierre via Schur Decomposition

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Abstract
In the paper the simplification of construction of nonlinear normal vibration modes by Shaw-Pierre in power series form is considered. The simplification can be obtained via change of variables in the equations of motion of dynamical system under consideration. This change of variables is constructed by means of so-called ordered Schur matrix decomposition. As the result of the transformation there is no need in solving nonlinear algebraic equations in order to evaluate coefficients of nonlinear normal mode.

Keywords
Nonlinear normal modes, matrix transformation, Schur decomposition

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Introduction

When investigation of the behavior of multi-dimensional dynamic systems takes place it is very importatnt to be able to construct reduced order model of that system. When a nonlinear system is studied, such reduction can be done, particularly, for regimes close to normal modes, since the system with many degrees of freedom (DOFs) behaves in such regimes as a single-DOF one. In such regimes all state space variables of the system change their values in a coherent manner.

Analytical dependencies which allow description of normal modes in mechanical systems can be obtained using two main approaches: by Kauderer-Rosenberg and by Shaw-Pierre. In the first case nonlinear normal mode (NNM) of a conservative (or close to conservative) mechanical system can be represented as a certain trajectory in the configuration space of the system [1]. The second concept which is applicable to non-conservative systems, was developed in works by S. Shaw, C. Pierre and their co-authors [2-4]. According to Shaw and Pierre, NNM of a non-conservative autonomous dynamical system can be defined as its invariant manifold. In this case all variables of the phase space of the system can be evaluated in unambiguous manner through a couple – certain displacement and corresponding velocity [4]:

$$\left\{q_i = q_i\left(q_m, s_m\right);\; s_i = s_i\left(q_m, s_m\right);\; \left(i = 1,..m-1, m+1,..N\right)\right. \tag{1}$$

where $q_i, s_i \; (i=\overline{1,N})$ are generalized displacements and velocities of the system.

Movement of the system in normal mode can be described as movement of representation point on the hypersurface (1).

There exists large number of works devoted to applications of NNMs. One can find here works devoted to vibration cancellation and energy pumping [5,6], papers devoted to vibrations of beams [7], plates and shells [8,9], vehicle suspension [10], rotordynamics [11-13], shallow arches etc. Comprehensive overview of different NNM theories and applications can be found in [1,14,15].

In the present work construction of NNMs is used together with matrix decomposition by I. Schur [16,17]. Schur transformation is a matrix similarity transformation. It allows one to transform a square matrix to an upper triangular one (using complex numbers) or to an upper quasi-triangular one (using real-valued matrices). Transformation of matrix to Schur form is an important method of eigenvalues calculation for non-symmetric matrices [17]. Also it is used for calculation of invariant subspaces of linear operators, for solving certain matrix equations (Sylvester matrix equation) etc.

In Section 1 of the pesent paper algorythm of NNM construction in power series form is discussed. Its is shown which peculiarities of equations of motion lead to nonlinear algebraic equations with respect to coefficients of NNM. In Section 2 some properties of Schur decomposition are described. In Section 3 application of Schur transformation to NNM construction is discussed. Two approaches are considered: conventional and modified Schur transformations. Section 4 contains an illustrative example.

1. Problem formulation

The present work is devoted to construction of NNMs by Shaw-Pierre in power series form. One of the problems that arise during this approach is that some of coefficients of power series that represent NNM must be evaluated from a system of nonlinear algebraic equations, and initial approximation for these coefficients is usually unknown.

Let us consider in brief the process of construction of NNMs by Shaw-Pierre in power series form according to [4] and find out possible causes of difficulties in computation. Consider autonomous non-conservative dynamical system (2):

$$\left\{\dot{q}_i = s_i;\;\; \dot{s}_i = f_i\left(\overline{q},\overline{s}\right);\right. \qquad \left(i=\overline{1,N}\right). \tag{2}$$

The functions $f_i\left(\overline{q},\overline{s}\right)$ are assumed to be analytical functions in the vicinity of zero equlibrium position. It is assumed that the system is free of internal resonances.

Let q_1 and s_1 be the independent variables for the considered NNM (1) (that is *m=1*). Let us denote $q_1 = u,\, s_1 = v$. Differentiation with respect to time t now becomes a partial differential operator:

$$\frac{\mathrm{d}}{\mathrm{d}t} = \dot{q}_1\frac{\partial}{\partial q_1} + \dot{s}_1\frac{\partial}{\partial s_1} = v\frac{\partial}{\partial u} + f_1\left(u,q_2,\ldots,v,s_2,\ldots\right)\frac{\partial}{\partial v} \tag{3}$$

By means of this operator the system of ODEs (2) is transformed to the following PDEs:

$$\begin{cases} v\dfrac{\partial q_k}{\partial u} + f_1\left(u,q_2,\ldots,v,s_2,\ldots\right)\dfrac{\partial q_k}{\partial v} = s_k, \\ v\dfrac{\partial s_k}{\partial u} + f_1\left(u,q_2,\ldots,v,s_2,\ldots\right)\dfrac{\partial s_k}{\partial v} = f_k\left(u,q_2,\ldots,v,s_2,\ldots\right). \end{cases} \qquad \left(k=\overline{2,N}\right) \tag{4}$$

Dependencies (1) are the solutions of equations (4). Such PDEs can be solved in different ways (the solution can be written in form of power series [3,4] if small/moderate oscillations are considered or found via Galerkin method [2] for large oscillations). Here the solution is found in power series form:

$$\begin{cases} q_n = \alpha_1^{(n)}u + \alpha_2^{(n)}v + \alpha_3^{(n)}u^2 + \alpha_4^{(n)}uv + \alpha_5^{(n)}v^2 + \ldots \\ s_n = \beta_1^{(n)}u + \beta_2^{(n)}v + \beta_3^{(n)}u^2 + \beta_4^{(n)}uv + \beta_5^{(n)}v^2 + \ldots \end{cases}, \; n=\overline{2,N} \tag{5}$$

Solution (5) is substituted into (4). At this stage the functions $f_k\left(\ldots\right)$ are considered to be polynomials (or they should be expanded in power series otherwise). When terms of the same power of u and v are equated in the obtained equalities, this leads to the recurrent system of algebraic equations with respect to unknown coefficients $\alpha_k^{(n)},\beta_k^{(n)}$. Among others there exists a closed subsystem of nonlinear equations with respect to $\alpha_1^{(n)},\alpha_2^{(n)},\beta_1^{(n)},\beta_2^{(n)}$, that is, the coefficients of linear terms in (5).

All other equations in the recurrent system are linear with respect to unknowns of current step but nonlinear with respect to quantities evaluated previously. That is, there can be found system of linear algebraic equations with respect to $\alpha_3^{(n)},\alpha_4^{(n)},\alpha_5^{(n)},\beta_3^{(n)},\beta_4^{(n)},\beta_5^{(n)}$ (coefficients by quadratic terms). Both its matrix and right hand side depend on previously evaluated $\alpha_1^{(n)},\alpha_2^{(n)},\beta_1^{(n)},\beta_2^{(n)}$. The same for coefficients by cubic terms and so on. This means that once $\alpha_1^{(n)},\alpha_2^{(n)},\beta_1^{(n)},\beta_2^{(n)}$ are found, all other coefficients are evaluated in *unique sequential way.*

It follows from the above considerations that the initial phase of calculation process (calculation of $\alpha_1^{(n)},\alpha_2^{(n)},\beta_1^{(n)},\beta_2^{(n)}$) is more difficult than others, because $\alpha_1^{(n)},\alpha_2^{(n)},\beta_1^{(n)},\beta_2^{(n)}$ are evaluated from *nonlinear algebraic equations*, and usually no initial approximation for these coefficients is provided.

Sometimes this problem may be overcome by introduction of some additional requirements. For example, one may search for such an invariant manifold (1) at which variables q_m, s_m have much larger amplitudes (active variables) than other variables of the phase space. In such case coefficients of series (5) are expected to be small and therefore one may use zero initial approximation for $\alpha_1^{(n)},\alpha_2^{(n)},\beta_1^{(n)},\beta_2^{(n)}$. Different NNMs can be found by choosing different pairs q_m, s_m as independent variables. This approach was used by the author in [11,12].

Nonlinearity of algebraic equations with respect to $\alpha_1^{(n)},\alpha_2^{(n)},\beta_1^{(n)},\beta_2^{(n)}$ is caused by the structure of the equations (4), namely, by terms of such sort: $f_1\left(u,q_2,...,v,s_2,...\right)\dfrac{\partial q_k}{\partial v}$. Indeed, let the function $f_1(...)$ be represented as

$$f_1\left(u,q_2,...,v,s_2,...\right)=a_1u+\sum_{n=2}^{N}a_nq_n+b_1v+\sum_{n=2}^{N}b_ns_n+\varphi_1\left(u,q_2,...,v,s_2,...\right) \tag{6}$$

where $\varphi_1(...)$ - is a polynomial of power 2 or higher.

Taking into account (5) and (6) one can obtain:

$$f_1\left(u,q_2,...,v,s_2,...\right)\frac{\partial q_k}{\partial v}=\left(a_1u+\sum_{n=2}^{N}a_n\left(\alpha_1^{(n)}u+\alpha_2^{(n)}v+...\right)+b_1v+\sum_{n=2}^{N}b_n\left(\beta_1^{(n)}u+\beta_2^{(n)}v+...\right)+...\right)\beta_2^{(k)} \tag{7}$$

Once parentheses are open in (7) the terms of type $\alpha_1^{(n)}\beta_1^{(k)}u$, $\alpha_1^{(n)}\beta_1^{(k)}v$, $\beta_1^{(n)}\beta_1^{(k)}u$ and $\beta_1^{(n)}\beta_1^{(k)}v$ arise, which leads to nonlinear algebraic equations with respect to $\alpha_1^{(n)},\alpha_2^{(n)},\beta_1^{(n)},\beta_2^{(n)}$.

Let us consider equations of motion (2) in matrix form:

$$\dot{\bar{y}}=[A]\bar{y}+\bar{\varphi}(\bar{y}), \tag{8}$$

where $\bar{y}=\{u,v,q_2,s_2,...\}^T$; vector-function $\bar{\varphi}(\bar{y})$ is purely nonlinear.

Nonlinear algebraic equations mentioned above do not appear if motion equations have the following form:

$$\begin{pmatrix}\dot{u}\\ \dot{v}\\ \dot{q}_2\\ \dot{s}_2\\ \vdots\end{pmatrix}=\begin{bmatrix}a_{11} & a_{12} & & B_1 & \\ a_{21} & a_{22} & & & \\ & & a_{33} & \cdots & a_{3,2N}\\ & B_2 & a_{43} & \cdots & a_{4,2N}\\ & & \vdots & \cdots & \vdots\end{bmatrix}\begin{pmatrix}u\\ v\\ q_2\\ s_2\\ \vdots\end{pmatrix}+\begin{pmatrix}\varphi_1\left(u,q_2,...,v,s_2,...\right)\\ \varphi_2\left(u,q_2,...,v,s_2,...\right)\\ \varphi_3\left(u,q_2,...,v,s_2,...\right)\\ \varphi_4\left(u,q_2,...,v,s_2,...\right)\\ \vdots\end{pmatrix} \tag{9}$$

If the block B_1 is filled with zeros then **no nonlinear algebraic equations with respect to** $\alpha_1^{(n)},\alpha_2^{(n)},\beta_1^{(n)},\beta_2^{(n)}$ **appear**. **They become linear instead**. On the other hand, if the block B_2 is filled with zeros then nonlinear **algebraic equations with respect to** $\alpha_1^{(n)},\alpha_2^{(n)},\beta_1^{(n)},\beta_2^{(n)}$ **do appear, but they have trivial solution**. So one of the invariant manifolds has zero linear part and thus can be computed much easier.

It will be shown further that each of these situations can be realized (but not simultaneously) if some specific change of variables is applied to the system (8). This change of variables can be constructed via so-called Schur matrix transformation. The transformation and its properties are discussed in the next Section.

2. Schur decomposition

▲ **Theorem 1 (real form of Schur decomposition) [16,17].** *For an arbitrary real-valued square matrix* $[A]$ *there exist an orthogonal matrix* $[Q]$ *such that* $[A]=[Q][T][Q]^T$ *where the upper quasi-triangular matrix* $[T]$ *has the following structure:*

$$[T]=\begin{bmatrix} T_1 & * & * & * \\ & T_2 & * & * \\ & & \ddots & * \\ 0 & & & T_m \end{bmatrix},$$

where block matrices T_i are blocks 1x1 or 2x2 corresponding to real eigenvalues and conjugate pairs of complex eigenvalues of the matrix $[A]$ respectively. The sequence of the diagonal blocks in the matrix $[T]$ may be arbitrary.

Correspondence between matrix blocks and eigenvalues should be understood as follows. Each 1x1 block contains some real eigenvalue whereas eigenvalues of each 2x2 block are included in the spectrum of the matrix $[A]$.

Construction of the matrix $[T]$ is performed iteratively via QR-algorithm [17]. So it is impossible to control the sequence $T_1, T_2, ...$ of diagonal blocks during this process. However, once it is constructed, it is possible to rebuild (reorder) this matrix so that first *k* diagonal blocks correspond to a certain subset of eigenvalues [18-20]. This is needed for construction of invariant subspaces of a linear operators and for finding bases in these subspaces.

It should be noted that Schur decomposition with reordering is available as a standard routine in some popular computational software (Matlab, Scilab, LAPACK), so the details of this procedure are not discussed here. However it should be noted that reordering is not possible if eigenvalues of reordered blocks are too close. This happens because matrix transformations become singular [18].

3. Application of Schur decomposition to construction of invariant manifold of quasi-linear mechanical system

Consider equations of motion of quasilinear dissipative mechanical system with *N* degrees of freedom in matrix form:

$$\dot{\bar{y}}=[A]\bar{y}+\bar{\Phi}(\bar{y}) \tag{10}$$

Vector $\bar{y}=\{x_1,...,x_N,\dot{x}_1,...,\dot{x}_N\}^T$ consists of generalized displacements and velocities of the system. The vector $\bar{\Phi}(\bar{y})$ - contains nonlinear analytical functions (polynomials of power greater than 1).

It is supposed that damping in the system is small and matrix $[A]$ has *N* pairs of complex-conjugated eigenvalues (non-multiple). Each pair can be assigned with corresponding invariant manifold represented as NNM by Shaw-Pierre.

Let us consider constriction of the NNM by Shaw-Pierre which correspond to the pair of eigenvalues $\{\lambda_k,\bar{\lambda}_k\}$.

3.1. Usage of conventional Schur decomposition with reordering

As the first step Schur decomposition with reordering for matrix $[A]$ of the system (10) is considered: $[A]=[Q][T][Q]^T$. Reordering should be performed in such way that eigenvalues of the starting diagonal block T_1 of matrix $[T]$ has eigenvalues $\{\lambda_k,\bar{\lambda}_k\}$. In this case matrix $[T]$ has the structure shown on Fig. 1.

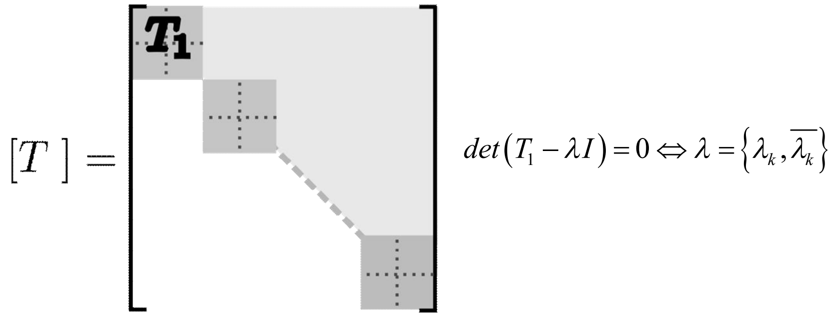


Figure 1. Structure of the Schur matrix. Nonzero elements are shaded in grey.

Now the following change of variables is introduced into (10): $\bar{y} = [Q]\bar{z}$. The new equations are multiplied from the left by $[Q]^{-1} = [Q]^T$. If one denotes $[Q]^T \bar{\Phi}([Q]\bar{z}) = \bar{F}(\bar{z})$, then the transformed equations have the form:

$$\dot{\bar{z}} = [T]\bar{z} + \bar{F}(\bar{z}) \tag{11}$$

Let us introduce the following notation: $z_1 = u,\ z_2 = v$; $\bar{z}_* = \{z_3, z_4, ..., z_{2N}\}^T$, $\bar{t}_{1*} = \{t_{13}, t_{14}, ..., t_{1,2N}\}^T$, $\bar{t}_{2*} = \{t_{23}, t_{24}, ..., t_{2,2N}\}^T$, $[t_{**}] = [t_{ij}]$ $\left(i, j = \overline{3, 2N}\right)$, $\bar{F}_* = \{F_3, F_4, ..., F_{2N}\}^T$. Using these equations (11) can be rewritten into

$$\begin{cases} \dot{u} = t_{11}u + t_{12}v + \bar{t}_{1*}^T\,\bar{z}_* + F_1(u, v, \bar{z}_*) \\ \dot{v} = t_{21}u + t_{22}v + \bar{t}_{2*}^T\,\bar{z}_* + F_2(u, v, \bar{z}_*) \\ \dot{\bar{z}}_* = \qquad\qquad [t_{**}]\bar{z}_* + \bar{F}_*(u, v, \bar{z}_*) \end{cases} \tag{12}$$

Now the change of independent variables is performed: $\dfrac{d}{dt} = \dot{u}\dfrac{\partial}{\partial u} + \dot{v}\dfrac{\partial}{\partial v}$. The NNM by Shaw-Pierre is introduced as

$$\bar{z}_* = \bar{\alpha}_{10}u + \bar{\alpha}_{01}v + \sum_{i+j\ge 2} \bar{\alpha}_{ij}u^i v^j \tag{13}$$

where vectors $\bar{\alpha}_{mn}$ are composed of unknown coefficients of the NNM. As the result the following equality is obtained:

$$\begin{aligned} &\left(\underline{\underline{\left(t_{11} + \bar{t}_{1*}^T\,\bar{\alpha}_{10}\right)u + \left(t_{12} + \bar{t}_{1*}^T\,\bar{\alpha}_{01}\right)v}} + (...)\right)\left(\underline{\underline{\bar{\alpha}_{10}}} + \frac{\partial}{\partial u}\left(\sum_{i+j\ge 2}\bar{\alpha}_{ij}u^i v^j\right)\right) + \\ &+\left(\underline{\underline{\left(t_{21} + \bar{t}_{2*}^T\,\bar{\alpha}_{10}\right)u + \left(t_{22} + \bar{t}_{2*}^T\,\bar{\alpha}_{01}\right)v}} + (...)\right)\left(\underline{\underline{\bar{\alpha}_{01}}} + \frac{\partial}{\partial v}\left(\sum_{i+j\ge 2}\bar{\alpha}_{ij}u^i v^j\right)\right) = \\ &= \underline{\underline{[t_{**}]\bar{\alpha}_{10}u + [t_{**}]\bar{\alpha}_{01}v}} + (...) \end{aligned} \tag{14}$$

Here $(\ldots)$ denotes terms of power greater than 1.

Algebraic equations for unknowns $\bar{\alpha}_{10}$ and $\bar{\alpha}_{01}$ can be obtained when underlined terms in (14) are used:

$$\begin{aligned} &\left(t_{11} + \bar{t}_{1*}^T\,\bar{\alpha}_{10}\right)\bar{\alpha}_{10} + \left(t_{21} + \bar{t}_{2*}^T\,\bar{\alpha}_{10}\right)\bar{\alpha}_{01} = [t_{**}]\bar{\alpha}_{10} \\ &\left(t_{12} + \bar{t}_{1*}^T\,\bar{\alpha}_{01}\right)\bar{\alpha}_{10} + \left(t_{22} + \bar{t}_{2*}^T\,\bar{\alpha}_{01}\right)\bar{\alpha}_{01} = [t_{**}]\bar{\alpha}_{01} \end{aligned} \tag{15}$$

This system (15) is nonlinear but it has trivial solution $\bar{\alpha}_{01} = \bar{\alpha}_{10} = 0$. This solution exactly corresponds to the manifold under consideration. Indeed, if system (11) was linear ($\dot{\bar{z}} = [T]\bar{z}$) then the manifold under consideration (corresponding to $\{\lambda_k, \bar{\lambda}_k\}$) would be $z_m = 0$ $\left(m = \overline{3, 2\text{N}}\right)$ due to the structure of matrix $[T]$. Since the linear part of NNM (13) remains the same both for linear and nonlinear case, the solution $\bar{\alpha}_{01} = \bar{\alpha}_{10} = 0$ is the sought-for one.

Therefore there is no need in composing and solving equations for $\bar{\alpha}_{10}$ and $\bar{\alpha}_{01}$. In order to determine the NNM one needs only to compute the coefficients of nonlinear terms of (13) which can be done as described in Section 1.

3.2. Usage of modified Schur decomposition (alternative approach)

At the beginning let us prove the theorem concerning modified form of Schur decomposition.

▲ **Theorem 2 (modified real form of Schur decomposition)** *For an arbitrary real-valued square matrix $[A]$ besides Schur expansion of form $[A] = [Q][T][Q]^T$ (where $[T]$ is an upper quasi-triangular matrix) there exists an alternative decomposition $[A] = [P][L][P]^T$ where $[L]$ - lower quasi-triangular matrix, and diagonal blocks of matrices $[T]$ and $[L]$ correspond to the spectrum of matrix $[A]$. Matrices $[Q]$ and $[P]$ are orthogonal ones.*

► Existence of the decomposition $[A]=[Q][T][Q]^T$ is guaranteed by *Theorem 1*. On the other hand similar decomposition exists for transposed matrix $[A]^T=[P][T'][P]^T$, here $[T']$ - is an upper quasi-triangular matrix and $[P]$ is an orthogonal one. If the latter equality is transposed, it follows from it that $[A]=\left([P]^T\right)^T[T']^T[P]^T=[P][T']^T[P]^T$. Denote $[L]=[T']^T$, therefore the decomposition now looks as $[A]=[P][L][P]^T$. Since $[T']$ is an upper quasi-triangular matrix, matrix $[L]=[T']^T$ is, obviously, a lower quasi-triangular one. Transposition does not change spectrum of matrix, therefore the spectrum of $[A]$ and $[A]^T$ is the same. This spectrum corresponds to diagonal blocks of both $[T]$ and $[T']$ according to *Theorem 1*. Since $[L]=[T']^T$, the diagonal blocks of $[L]$ also correspond to the spectrum of $[A]$. Q.E.D. ◄

If $[A]^T=[P][T'][P]^T$ is Schur decomposition with reordering, then the expansion $[A]=[P][L][P]^T$ can be built in such way that first *k* diagonal blocks of $[L]$ correspond to a given subset of eigenvalues (see below).

Let us consider constriction of the NNM by Shaw-Pierre which correspond to the pair of eigenvalues $\{\lambda_k,\bar{\lambda}_k\}$ in (10).

As the first step modified Schur decomposition with reordering for matrix $[A]$ of the system (10) is considered: $[A]=[P][L][P]^T$. This can be done in the following way. Firstly, Schur transformation with reordering is applied to the matrix $[A]^T$: $[A]^T=[P][T'][P]^T$. Reordering should be performed in such way that eigenvalues of the starting diagonal block T_1 of matrix $[T']$ has eigenvalues $\{\lambda_k,\bar{\lambda}_k\}$. Matrix $[L]$ is then obtained as $[L]=[T']^T$. In this case matrices $[T']$ and $[L]$ have the structure shown on Fig. 2.

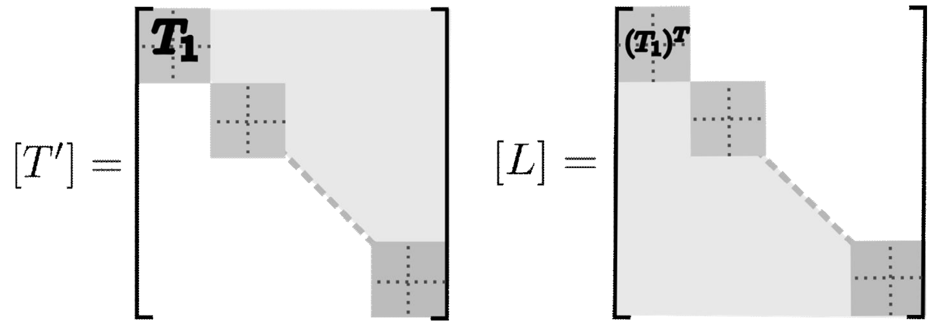


Figure 2. Structure of the Schur matrices in the alternative approach. Nonzero elements are shaded in grey.

Now the following change of variables is introduced into (10): $\bar{y}=[P]\bar{z}$. The new equations are multiplied from the left by $[P]^{-1}=[P]^T$. If one denotes $[P]^T\bar{\Phi}([P]\bar{z})=\bar{F}(\bar{z})$, then the transformed equations have the form:

$$\dot{\bar{z}}=[L]\bar{z}+\bar{F}(\bar{z}) \tag{16}$$

Let us introduce the following notation: $z_1=u,\ z_2=v$; $\bar{l}_{*1}=\{l_{31},l_{41},\ldots,l_{2N,1}\}^T$, $\bar{l}_{*2}=\{l_{32},l_{42},\ldots,l_{2N,2}\}^T$, $[l_{**}]=[l_{ij}]$ $\left(i,j=\overline{3,2N}\right)$, $\bar{F}_*=\{F_3,F_4,\ldots,F_{2N}\}^T$. Using these equations (16) can be rewritten into

$$\begin{cases}\dot{u}=l_{11}u+l_{12}v \qquad\qquad +F_1(u,v,\bar{z}_*)\\ \dot{v}=l_{21}u+l_{22}v \qquad\qquad +F_2(u,v,\bar{z}_*)\\ \dot{\bar{z}}_*=\bar{l}_{*1}u+\bar{l}_{*2}v+[l_{**}]\bar{z}_*+\bar{F}_*(u,v,\bar{z}_*)\end{cases} \tag{17}$$

Now the change of independent variables is performed: $\frac{d}{dt}=\dot{u}\frac{\partial}{\partial u}+\dot{v}\frac{\partial}{\partial v}$. The NNM by Shaw-Pierre is introduced as expansion (13). As the result the following equality is obtained:

$$\left(\underline{\underline{l_{11}u+l_{12}v}}+(\ldots)\right)\left(\underline{\underline{\bar{\alpha}_{10}}}+\frac{\partial}{\partial u}\left(\sum_{i+j\geq 2}\bar{\alpha}_{ij}u^iv^j\right)\right)+$$
$$+\left(\underline{\underline{l_{21}u+l_{22}v}}+(\ldots)\right)\left(\underline{\underline{\bar{\alpha}_{01}}}+\frac{\partial}{\partial v}\left(\sum_{i+j\geq 2}\bar{\alpha}_{ij}u^iv^j\right)\right)= \tag{18}$$
$$=\underline{\underline{\bar{l}_{*1}u+\bar{l}_{*2}v+[l_{**}]\bar{\alpha}_{10}u+[l_{**}]\bar{\alpha}_{01}v}}+(\ldots)$$

Here $(\ldots)$ denotes terms of power greater than 1.

Algebraic equations for unknowns $\bar{\alpha}_{10}$ and $\bar{\alpha}_{01}$ can be obtained using underlined terms in (18):

$$\begin{aligned} l_{11}\bar{\alpha}_{10}+l_{21}\bar{\alpha}_{01}&=\bar{l}_{*1}+[l_{**}]\bar{\alpha}_{10}\\ l_{12}\bar{\alpha}_{10}+l_{22}\bar{\alpha}_{01}&=\bar{l}_{*2}+[l_{**}]\bar{\alpha}_{01}\end{aligned} \tag{19}$$

System (19) is a system of linear algebraic equations, which allows one to easily compute unknowns $\bar{\alpha}_{10}$ and $\bar{\alpha}_{01}$. All other coefficients are obtained in conventional manner (see Section 1).

4. Example

Consider the nonlinear 3-DOF system depicted on Figure 3. Its motion equations have form (20).

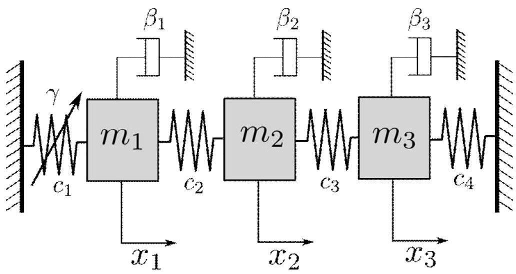


Figure 3. Three-DOF nonlinear system.

$$\begin{cases} m_1\ddot{x}_1+\beta\dot{x}_1+c_1x_1+c_2(x_1-x_2)+\gamma x_1^3=0\\ m_2\ddot{x}_2+\beta\dot{x}_2+c_2(x_2-x_1)+c_3(x_2-x_3)=0\\ m_3\ddot{x}_3+\beta\dot{x}_3+c_4x_3+c_3(x_3-x_2)=0\end{cases} \tag{20}$$

Parameters of the system are taken as follows: $m_1=2,\ m_2=0.5,\ m_3=1,\ c_1=c_2=c_3=c_4=1,\ \gamma=0.2,\ \beta=0.07$. Also equations (20) are subject of time scaling $\tau=\omega_1 t,\ \frac{d}{dt}=\omega_1\frac{d}{d\tau},\ \frac{d^2}{dt^2}=\omega_1^2\frac{d^2}{d\tau^2}$ where $\bar{\omega}=\{0.726062,\ 1.239920,\ 2.221583\}$ - eigenfrequencies. Thaking this into account, system (20) can be rewritten in standard form:

$$\begin{cases} \dot{y}_1=y_4;\quad \dot{y}_2=y_5;\quad \dot{y}_3=y_6;\\ \dot{y}_4=-1.89693y_1+0.94846y_2-0.048205y_4-0.18969y_1^3\\ \dot{y}_5=3.7938y_1-7.5877y_2+3.7938y_3-0.19282y_5\\ \dot{y}_6=1.8969y_2-3.7938y_3-0.09641y_6\end{cases} \tag{21}$$

Correctness of the presented approaches can be confirmed in the following way. First, equations (21) are integrated numerically. Initial point for numerical integration is taken on the surface of pre-calculated NNM. On the next step the trajectory obtained numerically (coordinates *y*) is translated to the coordinates in which NNM is defined (coordinates *z*). If the results are correct, the representation point (and the trajectory itself) must follow the surface of the NNM.

Eigenvalues of the matrix of linearized equations (21) are:

$$- 0.0389079 \pm 0.9995934I$$
$$- 0.0456418 \pm 1.7067491I$$
$$- 0.0841687 \pm 3.0582103I$$

For example, let us construct NNM corresponding to the first pair of values using both techniques from Section 3.1 and 3.2. Reordering of Schur matrices is performed in such way that the first one of diagonal blocks of the matrices $[T]$ and $[L]$ has eigenvalues close either to $+I$ or to $-I$. This was done using freeware Scilab software.

Approximation of the NNM found by means of conventional Schur decomposition:

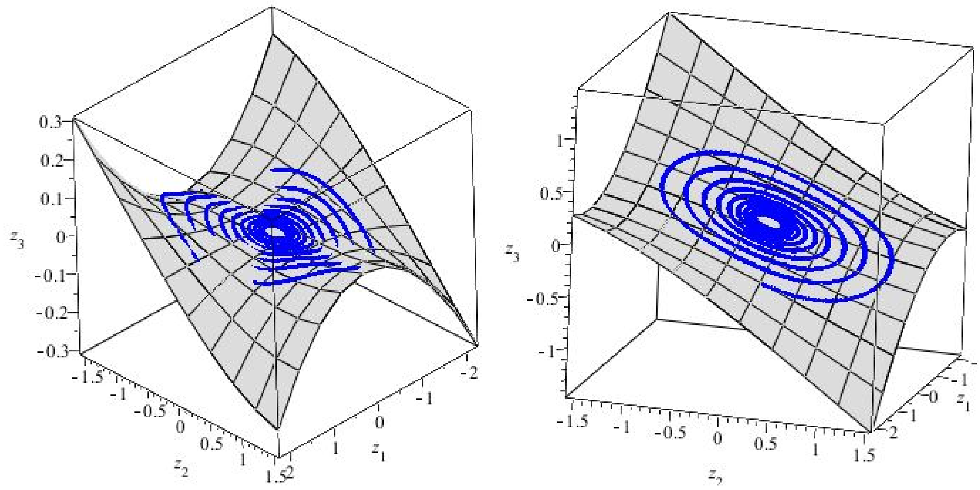
$$\begin{aligned} z_3 &= 0.01007 z_2^3 + 0.01142 z_1 z_2^2 - 0.03938 z_1^2 z_2 - 0.00243 z_1^3 \\ z_4 &= 0.01091 z_2^3 - 0.10694 z_1 z_2^2 - 0.03016 z_1^2 z_2 + 0.04069 z_1^3 \\ z_5 &= 0.00726 z_2^3 + 0.01662 z_1 z_2^2 + 0.02064 z_1^2 z_2 + 0.00558 z_1^3 \\ z_6 &= -0.00923 z_2^3 + 0.00691 z_1 z_2^2 - 0.00614 z_1^2 z_2 + 0.01203 z_1^3 \end{aligned} \tag{22}$$

found by means of modified Schur decomposition:

$$\begin{aligned} z_3 &= -0.23387 z_1 - 0.28916 z_2 + 0.01630 z_2^3 - 0.02458 z_1 z_2^2 - 0.05976 z_1^2 z_2 + 0.00567 z_1^3 \\ z_4 &= 0.27033 z_1 - 0.25449 z_2 - 0.02352 z_2^3 - 0.16776 z_1 z_2^2 + 0.06497 z_1^2 z_2 + 0.06121 z_1^3 \\ z_5 &= -0.31409 z_1 + 0.27348 z_2 + 0.01426 z_2^3 - 0.00136 z_1 z_2^2 + 0.02661 z_1^2 z_2 - 0.01850 z_1^3 \\ z_6 &= -0.24890 z_1 - 0.33618 z_2 - 0.01324 z_2^3 - 0.01370 z_1 z_2^2 - 0.01939 z_1^2 z_2 - 0.01790 z_1^3 \end{aligned} \tag{23}$$

(it should be noted here that in each case different coordinate transformations $\bar{y} -> \bar{z}$ were used)

On Figure 4 the trajectory of representation point of the system and NNM itself are shown. Therefore both dependencies (22) and (23) define invariant manifold (which is the same in both cases).



(a) (b)

Figure 4. Trajectory of representation point and invariant manifold (NNM) surfaces. (a) – conventional Schur decomposition is used, (b) - modified Schur decomposition is used

Conclusions

In the present work two ways of application of Schur matrix decomposition to NNM construction are considered. If NNM by Shaw-Pierre is constructed in power series form both of the discussed approaches allow one to overcome a major problem of the method – presence of nonlinear algebraic equations with respect to coefficients of linear part of the NNM. In one case Scur transformation allows one to avoid solving the algebraic equations as the sought-for solution is trivial one. The second approach allows one to introduce such change of variables that coefficients of linear part of the NNM are evaluated from a system of linear algebraic equations. Cost of such simplification is usage of computationally intensive algorythm of reordering of Schur matrices.

Also the presented approaches extend NNMs by Shaw-Pierre paradigm: during NNM construction independent variables in NNM expressions may not be the couple of variables of type "displacement + velocity".

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